Algebraic curves
Solutions sheet 10

May 22, 2024

Unless otherwise specified, k is an algebraically closed field.

Exercise 1. For $n, d \geq 1$, let $V(d,n)$ the k-vector space of forms of degree d in $k[X_1, \ldots, X_n]$.

1. Compute $\dim_k(V(d,n))$ for $d \geq 1$ and $n = 1, 2, 3$. Can you find a formula for arbitrary n?

Set $n = 2$. Let $L_i,\ i \geq 1$ and $M_j,\ j \geq 1$ be two sequences of non-zero linear forms in $k[X,Y]$ such that $L_i \neq \lambda M_j$ for all $i, j \geq 1$, $\lambda \in k$. Consider $A_{ij} = L_1 \ldots L_i M_1 \ldots M_j,\ i, j \geq 0$ (if $i = 0$ or $j = 0$, the empty product is taken as 1).

2. Show that $A_{ij},\ i + j = d,\ i, j \geq 0$ form a basis of $V(d, 2)$. (Hint: think of dehomogenizing the A_{ij} by setting $Y = 1$.)

Solution 1.

1. As we have already seen throughout those exercise sheets, we can compute $\dim_k(V(d,n))$ in a combinatorial way, as $\dim_k(V(d,n)) = \{d_1 + \cdots + d_n = d \mid d_k \in [\![1, n]\!]\} = \binom{n+d-1}{n-1}$.
2. We first see that the set of $A_{ij},\ i + j = d$ as cardinality $d + 1$, which is the same as $\dim_k(V(d, 2))$. Now, as for all $\lambda \in k$, all i, j, $L_i \neq \lambda M_j$, the deshomogenization $(L_i)_*$ and $(M_j)_*$ does not share any root. Thus, $L_1 \ldots L_d$, $L_1 \ldots L_{d-1} M_1$, $\ldots$, $M_1 \ldots M_d$ are linearly independant. (If there is a linear combination, take its deshomogenisation and evaluate at well-chosen roots to show that the coefficients are all zero).

Exercise 2. Recall properties 1 to 9 of intersection numbers from the course (Thm. 4.5). Prove property 8 using only properties 1 to 7. (Hint: introduce a uniformizer ϖ of $\mathcal{O}_P(F)$ and rewrite the factorization $G = u\varpi^n$, $u \in \mathcal{O}_P(F)^\times$ in terms of polynomials in $k[X,Y]$.)

Solution 2. We need to show that for P a simple point of F, $I(P, F \cap G) = ord_P^F(G)$. We reduce to $P = (0,0)$. We have 3 distinct cases :

- $G(P) \neq 0$. Then $I(P, F \cap G) = ord_P^F(G) = 0$. Indeed, $G \in \mathcal{O}_P(F)^*$.
- P lies in a common component of F and G. By 1), $I(P, F \cap G) = \infty$. Moreover, $G = 0$ in $\Gamma(F)$ since F is irreducible. So $G = 0$ in $\mathcal{O}_P(F)$, so $ord_P^F(G) = \infty$.

- Otherwise, let L be a line intersecting F transversally at P with $ord_P^F(L) = 1$. Let $n := ord_P^G(L)$. Hence $G = u \cdot L^n$ in $\mathcal{O}_P(F)$, where $u = \frac{f}{g}$, $f(P) \neq 0$, $g(P) \neq 0$. Now, in $\mathcal{O}_P(F)$

$$gG = fL^n$$

so in $k[X,Y]$, there exists h such that

$$gG = fL^n + hF$$

Using 1) and 6) we get $I(P, F \cap G) = I(P, F \cap gG)$. Using 7) and again 1) and 6), we get : $I(P, F \cap gG) = I(P, F \cap fL^n) = I(P, F \cap L^n) = nI(P, F \cap L) = n$.

Exercise 3. Compute the intersection numbers at $P = (0,0)$ of various pairs of the following curves:

- $A = Y - X^2$
- $B = Y^2 - X^3 + X$
- $C = Y^2 - X^3$
- $D = Y^2 - X^3 - X^2$
- $E = (X^2 + Y^2)^2 + 3X^2Y - Y^3$
- $F = (X^2 + Y^2)^3 - 4X^2Y^2$

Solution 3. We use the definition.

- $A \cap B$: $\mathcal{O}_P(\mathbb{A}^2)/(A,B) = \mathcal{O}_P(k[X,Y]/(Y - X^2, Y^2 - X^3 + X) = \mathcal{O}_P(k[X]/(X^4 - X^3 + X))$ Then, $X^3 + X^2 + 1$ is invertible in $\mathcal{O}_P(\mathbb{A}^2)$ so that

$$I(P, A \cap B) = dim_k(k[X]_{(X)}/X \times k[X]_{(X)}/(X^3 + X^2 + 1)) = dim_k k \times \{1\} = 1$$

- $A \cap C$:

$$k[X,Y]/(Y - X^2, Y^2 - X^3) = k[X]/(X^4 - X^3) = k[X]/X^3 \times k[X]/(X-1)$$

  $I(P, A \cap C) = 3$.

- $A \cap D$.

$$k[X,Y]/(Y - X^2, Y^2 - X^3 - X^2) \simeq k[X]/(X^4 - X^3 - X^2) \simeq k[X]/X^2 \times k[X]/(1 + X - X^2)$$

  $I(P, A \cap D) = 2$.

- $A \cap E$. $E(X, X^2) = 4X^4 - X^6 + \ldots$. $I(P, A \cap E) = 4$ for $char\ k \neq 2$, $I(P, A \cap E) = 6$ otherwise.

- $A \cap F$. $F(X, X^2) = -3X^6 + 3X^8 + 3X^10 + X^12$.

  For $char\ k \neq 3$, $I(P, A \cap F) = 6$. Otherwise, $I(P, A \cap F) = 12$.

- $B \cap C$

$$k[X,Y]/(Y^2 - X^3 + X, Y^2 - X^3) \simeq k[X]/(X)$$

  $I(P, B \cap C) = 1$

- $B \cap D$.

$$k[X,Y]/(Y^2 - X^3 + X, Y^2 - X^3 - X^2) \simeq k[X,Y]/(Y^2 - X^3 + X, X + X^2) \simeq k[Y]/Y^2$$

  since $X^2 = -X \Rightarrow X^3 = -X^2 = X$.

  $I(P, B \cap D) = 2$

- $B \cap E$. Since $B = Y^2 - X(1 + X^2)$, we see that $ord_P^B(Y) = 1$ and $ord_P^B(X) = 2$. Then $ord_P^B(E) = 3 = I(P, B \cap E)$.

- $B \cap F$. With the same reasoning, and $F = X^6 + 3X^4Y^2 + 3Y^4X^2 + Y^6 - 4X^2Y^2$ has order 6 in $\mathcal{O}_P(B)$.

  $I(P, B \cap F) = 6$

Exercise 4. Consider the affine curves $F = Y - X^2$ and $L = aY + bX + c$, where $a, b, c \in k$ and $(a, b) \neq (0, 0)$.

1. Compute the intersection points $P \subseteq F \cap L$ and their intersection numbers $I(P, F \cap L)$. Consider $s = \sum_P I(P, F \cap L)$. Give a necessary and sufficient condition for $s = 1$.

Let us identify $\mathbb{A}_k^2$ with the affine open subset $U_1 = \{x_1 \neq 0\} \subseteq \mathbb{P}_k^2$, where we use projective coordinates x_1, x_2, x_3. Consider $\overline{V}$ (resp. $\overline{L}$) the closure of $V(F) \subseteq U_1$ (resp. $V(L)$) in $\mathbb{P}_k^2$.

2. Assume that $s = 1$. Show that $\overline{V}$ and $\overline{L}$ admit another intersection point outside U_1 and that the intersection number (computed in the affine plane U_2 or U_3) is 1.

3. Same questions with $F = XY - 1$.

Solution 4.

1. $0 = Y - X^2 = aY + bX + c \Rightarrow aX^2 + bX + c = 0$. If $a \neq 0$, $b^2 - 4ac \neq 0$, there are two simple points. If $b^2 - 4ac = 0$ there is a double point.

   If $a = 0$ there is one simple point.

   Hence

$$s = 1 \iff a = 0$$

2. If $a = 0$, take homogenisation $YZ - X^2$, $bX + cZ$. Then deshomogenize in U_2, get $0 = Z - X^2 = bX + cZ \Rightarrow bX + cX^2 = 0$ which has 1 or 2 simple points, depending on c (since $b \neq 0$).

3. We have

$$k[X,Y]/(XY - 1, aY + bX + c) = k[X^{\pm 1}]/(bX^2 + cX + a)$$

So if $b \neq 0$, this ring is $2-$dimensional, and there are two simple points or one double points depending on the discriminant.

If $b = 0$, the ring is $1-$dimensional.

So again, $s = 0 \iff b = 0$.

Then, $\bar{V} = V_P(XY - Z^2)$ and $\bar{L} = V_P(aY + cZ)$. Deshomogenization in $\{Y \neq 0\}$ yields

$$k[x,z]/(x - z^2, a + cz) \simeq k[z]/(a + cz)$$

which has dimension 1.

Exercise 5. Let F be an affine plane curve. Let L be a line that is not a component of F. Suppose that $L = \{(a + tb, c + td),\ t \in k\}$. Define $G(T) = F(a + Tb, c + Td)$ and consider its factorization $G(T) = \epsilon \prod_i (T - \lambda_i)^{e_i}$ where the λ_i are distinct.

1. Show that there is a natural one-to-one correspondence between the λ_i and the points $P_i \in L \cap F$.

2. Show that, under this correspondence, $I(P_i, L \cap F) = e_i$. In particular, $\sum_i I(P_i, L \cap F) \leq \deg(F)$ (see for instance exercise 4).

Solution 5.

1. We have $t = \lambda_i \iff F(a + \lambda_i b, c + \lambda_i d) = 0$ so that $\{P_i = (a + \lambda_i b, c + \lambda_i d)\} = F \cap L$.

2. Let $F_L = dX - bY - ad + bc$ be the function defining L. $k[X,Y]/(F, F_L) = k[T]/G = \prod_i K[T]/(T - \lambda_i)^{e_i}$ so that $I(P_i, F \cap L) = e_i$. We used a ring isomorphism given by $X \mapsto a + bT$, $Y \mapsto c + dT$.